ON THE CHARACTER OF STRONG JUMPS IN CERTAIN COMPLEX MEDIA

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Shock waves have been the subject of many investigations; of these, we only mention papers [1 to 5].

The present paper amplifies on the basic ideas proposed in [6] for the investigation of the structure of shock waves in a viscous gas. It appears that for a hypoelastic medium, within small strain accuracy, all discontinuous quantities inside a shock wave layer, vary in a similar manner. This circumstance permits the investigation of shock wave propagation in a hypoelstic medium.

The equations thus obtained are used to study the structure of a transverse shock wave in a Kelvin medium.

1. In order to investigate the structure of shock waves, we introduce a moving coordinate system (x_1, x_2, x_3) whose origin lies on some middle surface Σ, which is located within the shock wave layer with the (x_1, x_2) plane tangent to that surface.

Utilizing the symbol δ to denote differentiation with respect to time, the equations of conservation of mass and momentum, respectively, take the form

$$\{\rho(v_3 - G)\}_{,3} + (\rho v_\alpha)_{,\alpha} + \frac{\delta\rho}{\delta t} = 0 \qquad (1.1)$$

$$\sigma_{i3,3} + \sigma_{i\alpha,\alpha} = \rho(v_3 - G)v_{i,3} + \rho v_\alpha v_{i,\alpha} + \rho\frac{\delta v_i}{\delta t} \qquad (1.2)$$

Within the shock wave layer, the first term in the left-hand side of (1.1) and the first terms in the left- and right-hand sides of (1.2) are large in comparison with the remaining terms.

In the case of unsteady flow, the middle surface Σ will move with velocity G. The distance from the leading shock front surface to Σ will be denoted by h^+, and that from the trailing shock front surface to Σ, by h^-. Hence, the sum $(h^+ + h^-) = h$ is the shock layer thickness. It is assumed that the position of the surface Σ at any given time is known from nonviscous flow considerations. As the viscosity goes to zero, both shock fronts approach to the surface Σ.

The various discontinuous functions generally undergo their basic changes at different distances from Σ . Therefore, each discontinuous function has its own corresponding shock layer thickness. We denote the shock layer thicknesses for ρ, v_i and σ_{ij} by $h^{\pm}$, $h_i^{\pm}$ and $h_{ij}^{\pm}$, respectively, the full thicknesses being given by

$$h^+ + h^- = h, \quad h_i^+ + h_i^- = h_i, \quad h_{ij}^+ + h_{ij}^- = h_{ij} \tag{1.3}$$

Let ρ^*, v_i^* and σ_{ij}^* be, respectively, the density, velocity and stresses on Σ . Then the thicknesses of the shock layer are given by

$$\rho^*_{,3} = \pm \frac{\rho^{\pm} - \rho^*}{h^{\pm}}, \quad v^*_{i,3} = \pm \frac{v_i^{\pm} - v_i^*}{h_i^{\pm}}, \quad \sigma^*_{ij,3} = \pm \frac{\sigma_{ij}^{\pm} - \sigma_{ij}^*}{h_{ij}^{\pm}} \tag{1.4}$$

Here it is assumed that the gradients of the various functions within the shock wave are large while the corresponding thicknesses are small. It should be noted that this is not the only way of specifying the shock layer thickness.

Certain relations may be established among the various shock layer thicknesses. Thus, neglecting higher order quantities in (1.1) and (1.2), we obtain

$$\rho_{,3}(v_3 - G) + \rho v_{3,3} = 0, \qquad \sigma_{i3,3} = \rho (v_3 - G) v_{i,3} \tag{1.5}$$

To (1.5), we adjoin the results of integrating these equations across the shock layer

$$\rho (v_3 - G) = C, \qquad \sigma_{i3} = C v_i + C_i \tag{1.6}$$

Substituting (1.4) into (1.5) and utilizing (1.6), we obtain

$$h_3^{\pm} = h^{\pm}\rho^* / \rho^{\pm}, \qquad h_i^{\pm} = h_{i3}^{\pm} \tag{1.7}$$

For high density gases, liquids and other rheological substances, there is little variation in density across the shock wave, so that (1.7) yields $h_3^{\pm} \approx h^{\pm}$. For an ideal gas, $\sigma_{11} = \sigma_{22} = \sigma_{33}$ and $\sigma_{\alpha 3} = 0$, so that $h_{11}^{\pm} = h_{22}^{\pm} = h_{33}^{\pm}$, and (1.7) yields

$$h^{\pm} \approx h_3^{\pm} = h_{11}^{\pm} = h_{22}^{\pm} = h_{33}^{\pm} \tag{1.8}$$

Thus, for shock wave propagation in an ideal gas, the basic change in all discontinuous quantities takes place at the same distance from Σ .

Hereinafter we will need certain relations between thicknesses for discontinuous functions f_j and φ_i which are linearly interrelated by

$$\varphi_i = A_{ij} f_j + B_i, \qquad |A_{ij}| \neq 0 \tag{1.9}$$

The quantities A_{ij} and B_i are independent of x_3 .

Lemma 1. If the shock layer thicknesses h^+ and h^- are the same for all functions f_j, then the thicknesses h_i^+ and h_i^- for all functions φ_i will also be h^+ and h^-, respectively.

Lemma 2. If all functions f_j vary in a similar manner within the shock layer, i.e. if $f_j - f_j^+ = \nu[f_j]$, then the functions φ_i also vary in a similar manner inside the shock layer, with $\varphi_i - \varphi_i^+ = \nu[\varphi_i]$, and the same shock layer thickness is obtained for all the functions f_j and φ_i

$$h = 1 / \nu_{,3}^*, \qquad h^+ = -\nu^* / \nu_{,3}^*, \qquad h^- = (1 + \nu^*) / \nu_{,3}^* \tag{1.10}$$

From these Lemmas it follows that, within small strain accuracy, all discontinuous quantities inside a shock layer propagating in an elastic medium, vary in a similar manner, and have the same thicknesses.

In order to obtain the basic equations for the purpose of studying the shock wave structure, Equations (1.1) and (1.2) are integrated with respect to x_3. Taking into account (1.7), we obtain

$$\rho (v_3 - G) = C - \varphi, \qquad C = \rho^+ (v_3^+ - G) \tag{1.11}$$

$$\varphi = \int_{h^+}^{x_3} \left\{(\rho v_\alpha)_{,\alpha} + \frac{\delta\rho}{\delta t}\right\} dx_3, \qquad -h^- \leqslant x_3 \leqslant h^+ \tag{1.12}$$

$$\sigma_{i3} - Cv_i = C_i - \varphi_i, \qquad C_i = \sigma_{i3}^+ - Cv_i^+ \tag{1.13}$$

$$\varphi_i = \int_{h_i^+}^{x_3} \left\{\sigma_{i\alpha,\alpha} - \rho v_\alpha v_{i,\alpha} - \rho \frac{\delta v_i}{\delta t}\right\} dx_3 + \int_{v_i^+}^{v_i} \varphi dv_i, \quad -h_i^- \leqslant x_3 \leqslant h_i^- \tag{1.14}$$

The integrand in (1.10) is finite, and the interval of integration is small, so that φ is small. For the same reason, the first integral (1.12) is small. The second integral in (1.12) is small, because φ is small. Thus, φ and φ_i are small functions defined within the shock layer.

Noting that, for a shock wave of zero thickness, the dynamic conditions for the discontinuities [7] are given by

$$[\rho (v_3 - G)] = 0, \qquad [\sigma_{i3} - Cv_i] = 0 \tag{1.15}$$

we may consider the immediately preceding equations to be a first approximation for shock waves of small thickness. Hence, we conclude from (1.11) and (1.13) that φ and φ_i vanish on both fronts of the shock layer. Assuming that these functions are nonzero everywhere within the shock layer, we may approximate these functions by parabolas, obtaining

$$\varphi \approx \frac{A}{2h} (x_3 - h^+)(x_3 + h^-), \qquad A = \left[(\rho v_\alpha)_{,\alpha} + \frac{\delta\rho}{\delta t}\right] \tag{1.16}$$

$$\varphi_i \approx \frac{A_i}{2h_i} (x_3 - h_i^+)(x_3 + h_i^-)$$

$$A_i = \left[\sigma_{i\alpha,\alpha} - \rho v_\alpha v_{i,\alpha} - \rho \frac{\delta v_i}{\delta t}\right] + \left\{(\rho^* v_\alpha^*)_{,\alpha} + \frac{\delta\rho^*}{\delta t}\right] [v_i] \tag{1.17}$$

The quantities φ and φ_i become identically zero for one-dimensional steady flow. These small functions make an essential contribution for a shock wave with small discontinuities, when their values in (1.11) and (1.13) increase.

For h, h_i and $h_{ij} \to 0$, (1.11) and (1.13) become (1.16), provided φ and $\varphi_i \to 0$. To satisfy this condition, it is sufficient that the integrands in (1.12) and (1.14) be finite everywhere inside the shock layer, which in turn requires the boundedness of:

a) density

b) tangential velocity components on Σ

c) stresses on any element of area lying on Σ, with arbitrary normal.

For one-dimensional flow, two of the preceding conditions are satisfied independently of the properties of the medium. In that case, if the constitutive equations do not preclude the possibility of the existence of shock waves of zero thickness, then the propagation of such waves is possible.

In the solution of the viscous flow problem within the shock layer, it is assumed that the values of ρ, v_i and σ_{ij} on the leading and trailing shock fronts, respectively, coincide with the values of these quantities in front and behind the shock wave as obtained from the nonviscous flow problem; thus, the viscous effects are zero on both fronts. Consequently, the problem concerning the structure of the shock wave is reduced to the determination of viscous effects as a function of position within the shock wave, these effects vanishing at both shock fronts where $x_3 = \pm h^{\mp}$.

If φ, φ_i and nonlinear terms are neglected in (1.11),(1.13) and in the constitutive equations of the Kelvin medium, we find that the viscous effects vanish identically within the shock layer. Then the problem of the structure of the shock wave becomes meaningless. Therefore, it is necessary, in formulating the problem on the shock wave structure, to include the nonlinear terms in the equations pertaining to the structure of the shock layer. Hence it follows that the thickness and structure of a shock wave in a Kelvin medium are second order effects.

2. In hypoelastic, elasto-plastic and many other media it is impossible to determine the speed of propagation of shock waves [8]. We will show for the hypoelastic medium that by adjoining a viscous element in parallel with the hypoelastic element and utilizing the theory on shock wave structure the indeterminacy can be removed.

The constitutive equations for a hypoelastic medium may be written in the form [9]

$$\frac{D\sigma_{ij}}{Dt} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \qquad 2\varepsilon_{ij} = v_{i,j} + v_{j,i} \qquad (2.1)$$

Following Jaumann, covariant differentiation with respect to time in a moving coordinate system yields

$$\frac{D\sigma_{ij}}{Dt} = (v_3 - G)\,\sigma_{ij,3} + v_\alpha\sigma_{ij,\alpha} + \frac{1}{2}\sigma_{kj}(v_{k,i} - v_{i,k}) + \frac{1}{2}\sigma_{ki}(v_{k,j} - v_{j,k}) \quad (2.2)$$

Dividing (2.1) by $(v_3 - G)$ and integrating with respect to x_3 from the trailing wave front to the leading one, we obtain

$$[\sigma_{ij}] + \frac{\delta_{i3}}{2}P_{kjk} + \frac{\delta_{j3}}{2}P_{kik} - \frac{1}{2}(P_{3ij} + P_{3ji}) = \lambda J_3\delta_{ij} + \mu(\delta_{j3}J_i + \delta_{i3}J_j) \quad (2.3)$$

Here

$$P_{ijk} = \int_{v_k^-}^{v_k^+}\frac{\sigma_{ij}dv_k}{v_3 - G}, \qquad J_i = \int_{v_i^-}^{v_i^+}\frac{dv_i}{v_3 - G} \qquad (2.4)$$

If nonlinear terms in (2.1) are neglected, then we must set $P_{ijk} = 0$ in (2.3). In that case, we obtain the same shock waves as in a linear elastic medium. If the influence of nonlinear terms in (2.1) is to be taken into account, then P_{ijk} and J_α must be evaluated. In that case, the initial assumption is made that the flow taking place inside the shock wave is viscous, the corresponding rheological model consisting of a combination of hypoelastic element and a viscous element in parallel.

Upon solving the problem concerning shock wave structure, integration of (2.4) yields P_{ijk} and J_α. Then, letting the coefficient of viscosity go to zero, we obtain the limiting values of the above quantities, the shock wave thickness again becoming zero.

Completing the investigation of the shock wave structure and neglecting nonlinear terms in the constitutive equations (2.1), then utilizing Lemma 2, we obtain

$$\frac{v_i - v_i^+}{[v_i]} = \frac{\sigma_{ij} - \sigma_{ij}^+}{[\sigma_{ij}]} = \nu(x_k, t), \qquad -1 \leqslant \nu \leqslant 0 \tag{2.5}$$

Substituting (2.5) into (2.4) for $[v_3] \neq 0$, we obtain

$$J_i = \frac{[v_i]}{[v_3]} \ln \frac{v_3^+ - G}{v_3^- - G}, \qquad P_{ijk} = \frac{[\sigma_{ij}]\,[v_k]}{[v_3]} + \frac{\sigma_{ij}^-(v_3^+ - G) - \sigma_{ij}^+(v_3^- - G)}{[v_3]} J_k \tag{2.6}$$

If $[v_3] = 0$, then (2.4) yields

$$J_i = \frac{[v_i]}{v_3 - G}, \qquad P_{ijk} = \sigma_{ij}^* J_k, \qquad \sigma_{ij}^* = \frac{1}{2}(\sigma_{ij}^+ + \sigma_{ij}^-) \tag{2.7}$$

Consider propagation of the longitudinal shock wave for $[v_\alpha] = 0$ and for $[v_3] \neq 0$. In that case, $P_{ij\alpha} = J_\alpha = 0$, and consequently, the rotation of the medium does not affect the shock waves propagation. Solving (1.15) together with (2.3) and (2.6), we obtain

$$[\sigma_{i\alpha}] = 0, \quad \ln y = \frac{V}{y-1}, \quad y = \frac{G - v_3^+}{G - v_3^-}, \quad V = \frac{\rho^- [v_3]^2}{\lambda + 2\mu} \tag{2.8}$$

For given $v_3^\pm$ and ρ^-, (2.8) yields two values, y_1 and y_2, corresponding to two possible propagation speeds of longitudinal shock wave in hypoelastic media. It is readily established that $y_1 \geq 1$, $y_2 \leq 1$. A study of these inequalities reveals that the first condition is realized if v_3^- is between v_3^+ and G, while the second condition is realized if v_3^+ is between v_3^- and G. Analysis of (2.8) also shows that either v_3^+ and v_3^- are both greater or they are both smaller than G.

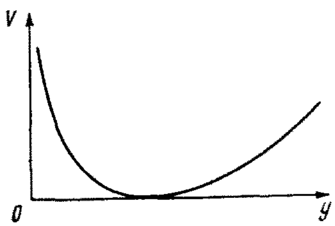

Fig. 1

The relationship between shock wave speed and jump magnitude is shown in Fig.1, where V and y are as given in (2.8).

For transverse shock wave propagation, Equations (1.15), (2.3) and (2.7) lead to

$$\{\sigma_{\alpha\beta}^* - (\sigma_{33}^* - 2\rho\,(v_3 - G)^2 + 2\mu)\,\delta_{\alpha\beta}\}\,[v_\beta] = 0 \tag{2.9}$$

Setting the determinant of this system equal to zero, we obtain (2.10)

$$2\rho\,(G - v_3)^2 = 2\mu - (\sigma_{11}^* + \sigma_{22}^* - 2\sigma_{33}^*) \pm \{(\sigma_{11}^* - \sigma_{22}^*)^2 + (2\sigma_{12}^*)^2\}^{1/2}$$

It follows from (2.10) that transverse shock waves may propagate in a hypoelastic medium with two speeds which are close to the speed of transverse sound waves and independent of the hydrostatic pressure.

3. Let us examine the structure of a transverse shock wave in a Kelvin medium [10], for which $[u_{3,3}] = 0$. The defining equations take the form

$$\sigma_{ij} = (\lambda e_{kk} + \xi \varepsilon_{kk}) \delta_{ij} + 2\mu e_{ij} + 2\eta \varepsilon_{ij} \qquad (3.1)$$

The Almansi tensor of finite deformation is given by [9]

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{k,i} u_{k,j} \qquad (3.2)$$

The coordinate system in the $x_1 x_2$ plane is oriented so as to make $[u_{2,3}] = 0$. The expression for the speed in terms of the distortion tensor may be obtained from

$$v_i = \frac{\delta u_i}{\delta t} - G u_{i,3} + v_k u_{i,k} \qquad (3.3)$$

Substituting (3.1) into (1.13) and making use of (3.2) and (3.3), we obtain

$$\eta v_{1,3} = B\ (u_{1,3} - u_{1,3}^{+})\ (u_{1,3} - u_{1,3}^{-}) - \varphi v_1 - \varphi_1 \qquad (3.4)$$

$$B = \frac{\mu}{\Delta} (1 - u_{1,1}) \{u_{2,1} u_{3,2} + u_{3,1} (1 - u_{2,2})\}$$

Here Δ is the determinant of the system of Equations (3.3).

For the steady case, (3.4) yields a quadratic equation from which the shock layer thickness h may be determined

$$\alpha h^2 + B\ [u_{1,3}]^2 h - \gamma\ [u_{1,3}] = 0 \qquad (3.5)$$

Here (3.6)

$$\alpha = A\ (G - v_3)\ u_{1,3}{}^* - A_1, \qquad \gamma = 4\eta\ (G - v_3), \qquad A = \rho\ (v_3 - G)\ [u_{1,31}]$$

$$A_1 = -\frac{1}{2} \rho\ (G - v_3)^2 [u_{1,3}^2]_{,1} + \rho\ (G - v_3)\ v_2\ [u_{1,32}] - \rho v^*_{\alpha,\alpha} (G - v_3)\ [u_{1,3}]$$

Of the two roots obtained, h_1 and h_2 , the positive is to be chosen.

Integrating (3.4), we obtain the variation of v_1 across the shock layer, while the variation of σ_{13} as function of x_3 is obtained from (3.1). The remaining velocity and stress components are continuous within small strain accuracy. The effect of the small quantities φ and φ_1 on the shock layer thickness in (3.5) is indicated by the coefficient α .

The propagation and structure of shock waves in elastoplastic media may be investigated in a similar manner, by adjoining various viscous elements to the rheological model.

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